A remark on Kac–Wakimoto hierarchies of D-type

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Abstract
For the Kac–Wakimoto hierarchy constructed from the principal vertex operator
realization of the basic representation of the affine Lie algebra $D_n^{(1)}$, we compute
the coefficients of the corresponding Hirota bilinear equations, and verify the
coincidence of these bilinear equations with the ones that are satisfied by
Givental's total descendant potential of the D_n singularity, as conjectured by
Givental and Milanov (2005 Simple singularities and integrable hierarchies *The
Breadth of Symplectic and Poisson Geometry* (*Prog. Math.* vol 232) (Boston:
Birkhäuser) pp 173–201).

PACS number: 02.20.Qs

1. Introduction

The theory on representation of theoretical aspects of soliton equations developed by Date,
Jimbo, Kashiwara, Miwa [1–4, 16] and Kac, Wakimoto [17, 18] plays a significant role in
several research areas of modern mathematical physics. For each affine Lie algebra $\mathfrak{g}$, together
with an integrable highest weight representation V of $\mathfrak{g}$ and a vertex operator construction R
of V, Kac and Wakimoto formulated a hierarchy of soliton equations. These equations can be
written down in terms of Hirota bilinear equations and their super analogue [18]. When $\mathfrak{g}$ is
the untwisted affinization of a simply laced finite Lie algebra, the Kac–Wakimoto hierarchy
coincides with the corresponding generalized Drinfeld–Sokolov hierarchy defined by Groot,
Hollowood and Miramontes [14, 15]. In particular, if the highest weight representation is the
basic one, and the vertex operator realization is constructed from the principal Heisenberg
subalgebra, then the Kac–Wakimoto hierarchy is equivalent to that of the Drinfeld–Sokolov
hierarchy associated with $\mathfrak{g}$ and the vertex c_0 of its Dynkin diagram [5].

In [10, 11], Givental constructed the total descendant potential for any semisimple
Frobenius manifold [6]. This potential is supposed to satisfy the axioms dictated by the
Gromov–Witten theory, such as the string equation, dilaton equation, topological recursion
relations and Virasoro constraints. Recently, Givental and Milanov [12, 13] showed that the

total descendant potentials for semisimple Frobenius manifolds associated with the simple singularities satisfy certain Hirota bilinear (quadratic) equations, and proved that for the A_n, D_4 and E_6 singularities these equations are equivalent to the corresponding Kac–Wakimoto hierarchies. They also conjectured that this fact is true for all simple singularities.

In this paper we compute explicitly the coefficients of the Kac–Wakimoto hierarchy constructed from the principal vertex operator realization of the basic representation of the affine Lie algebra $D_n^{(1)}$, while these coefficients are implicitly defined in [18], except for the case $n = 4$. This computation verifies Givental and Milanov's conjecture for the D_n singularity.

2. Kac–Wakimoto hierarchies of ADE type

Let $\mathfrak{g}$ be an untwisted affine Lie algebra of ADE-type, with rank n, Coxeter number h and normalized invariant bilinear form $(\cdot|\cdot)$. The set of simple roots and simple coroots are denoted by $\{\alpha_i\}_{i=0}^n$ and $\{\alpha_i^\vee\}_{i=0}^n$, respectively.

We denote the principal gradation of $\mathfrak{g}$ as $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$. The Cartan subalgebra of $\mathfrak{g}$, i.e. the 0-component $\mathfrak{g}_0$, has the following two decompositions:

$$\mathfrak{g}_0 = \mathring{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d = \bar{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Here, on the one hand, $\mathring{\mathfrak{h}} = \sum_{i=1}^n \mathbb{C}\alpha_i^\vee$, c is the central element and d is determined by the constraint

$$(\mathring{\mathfrak{h}}|d) = 0, \qquad (c|d) = 1, \qquad (d|d) = 0;$$

on the other hand, the subspace $\bar{\mathfrak{h}}$ is so chosen that the difference of the projections of any $x \in \mathfrak{g}_0$ onto $\mathring{\mathfrak{h}}$ and $\bar{\mathfrak{h}}$ is given by $\mathring{x} - \bar{x} = h^{-1}(\mathring{\rho}^\vee|\mathring{x})c$, where $\mathring{\rho}^\vee$ is an element of $\mathring{\mathfrak{h}}$ defined by the condition

$$\langle \alpha_i, \mathring{\rho}^\vee \rangle = 1, \qquad i = 1, \dots, n. \tag{2.1}$$

Let E be the set of exponents of $\mathfrak{g}$. For each $j \in E$ there exists $H_j \in \mathfrak{g}_j$ satisfying

$$(H_i|H_j) = h\,\delta_{i,-j}, \qquad [H_i, H_j] = i\,\delta_{i,-j}\, c. \tag{2.2}$$

They generate the principal Heisenberg subalgebra $\mathfrak{s} = \mathbb{C}c + \sum_{j\in E} \mathbb{C}H_j$.

In Kac and Wakimoto's construction of the hierarchies, it is essential to choose two bases $\{v_i\}$, $\{v^i\}$ of $\mathfrak{g}$ that are dual to each other. These two bases read

$$\{v_i\}: \quad \frac{1}{\sqrt{h}} H_j\,(j \in E), \qquad X_m^{(r)}(1 \leqslant r \leqslant n; m \in \mathbb{Z}), c, d; \tag{2.3}$$

$$\{v^i\}: \quad \frac{1}{\sqrt{h}} H_{-j}\,(j \in E), \qquad Y_{-m}^{(r)}(1 \leqslant r \leqslant n; m \in \mathbb{Z}), d, c, \tag{2.4}$$

such that

$$\left\{X_0^{(r)}\right\}_{r=1}^n, \left\{Y_0^{(r)}\right\}_{r=1}^n \text{ are two bases of } \bar{\mathfrak{h}}, \tag{2.5}$$

$$\left[H_j, X_m^{(r)}\right] = \beta_{r,\bar{j}} X_{m+j}^{(r)}, \qquad \left[H_j, Y_{-m}^{(r)}\right] = -\beta_{r,\bar{j}} Y_{-m+j}^{(r)}, \tag{2.6}$$

$$\left(X_l^{(r)}\big|Y_{-m}^{(s)}\right) = \delta_{r,s}\delta_{l,m}, \tag{2.7}$$

where $0 < \bar{j} < h$ is the remainder of j modulo h, and $\beta_{r,\bar{j}}$ are some complex numbers which depend on the choice of the two bases of $\mathfrak{g}$.

Let E_+ be the set of positive exponents. A representation of the Heisenberg subalgebra $\mathfrak{s}$ on the Fock space $\mathbb{C}[t_j; \ j \in E_+]$ is given by

$$c \mapsto 1, \qquad H_j \mapsto \frac{\partial}{\partial t_j}, \qquad H_{-j} \mapsto jt_j, \quad j \in E_+.$$

This can be lifted to a basic representation $L(\Lambda_0)$ of $\mathfrak{g}$ as follows:

$$\sum_{m\in\mathbb{Z}} X_m^{(r)} z^{-m} \mapsto -h^{-1}\big(\mathring{\rho}^\vee \big| \mathring{X}_0^{(r)}\big) X^{(r)}(t; z),$$

$$\sum_{m\in\mathbb{Z}} Y_{-m}^{(r)} z^m \mapsto -h^{-1}\big(\mathring{\rho}^\vee \big| \mathring{Y}_0^{(r)}\big) X^{(r)}(-t; z),$$

$$d_0 := hd + \mathring{\rho}^\vee \mapsto -\sum_{j\in E_+} jt_j \frac{\partial}{\partial t_j},$$

where $X^{(r)}(t; z)$ $(1 \leqslant r \leqslant n)$ are the vertex operators

$$X^{(r)}(t; z) = \left(\exp \sum_{j\in E_+} \beta_{r,\bar{j}} t_j z^j\right)\left(\exp - \sum_{j\in E_+} \frac{\beta_{r,\overline{-j}}}{jz^j} \frac{\partial}{\partial t_j}\right).$$

Such a realization of the basic representation $L(\Lambda_0)$ is called the *principal vertex operator construction*, see [17, 18] for details.

Theorem 2.1 [18]. *Consider the basic representation of a simply laced affine Lie algebra $\mathfrak{g}$ on the Fock space $L(\Lambda_0) = \mathbb{C}[t_j; \ j \in E_+]$ constructed as above. Denote by G the Lie group of the derived algebra $\mathfrak{g}'$ of $\mathfrak{g}$. A nonzero $\tau \in L(\Lambda_0)$ lies in the orbit $G \cdot 1$ if and only if τ satisfies the following hierarchy of Hirota bilinear equations:*

$$\left(-2h \sum_{j\in E_+} j\, y_j D_j + \sum_{r=1}^n g_r \sum_{m\geqslant 1} S_m^E(2\beta_{r,\bar{j}} y_j) S_m^E\left(-\frac{\beta_{r,\overline{-j}}}{j} D_j\right)\right)\left(\exp \sum_{j\in E_+} y_j D_j\right) \tau \cdot \tau = 0. \tag{2.8}$$

Here $g_r = \big(\mathring{\rho}^\vee \big| \mathring{X}_0^{(r)}\big)\big(\mathring{\rho}^\vee \big| \mathring{Y}_0^{(r)}\big)$, S_m^E are the elementary Schur polynomials of $\mathfrak{g}$ defined by $\exp \sum_{j\in E_+} y_j z^j = \sum_{m\geqslant 0} S_m^E(y_j) z^m$ and D_j are the Hirota bilinear operators defined by $D_j f \cdot g = \frac{\partial}{\partial u}\big|_{u=0} f(t_j + u) g(t_j - u)$.

Kac and Wakimoto explicitly gave the coefficients g_r, $\beta_{r,j}$ for the affine Lie algebras $A_n^{(1)}$, $D_4^{(1)}$ and $E_6^{(1)}$ in [18], however, these coefficients remain implicit for other affine Lie algebras. We proceed to compute them for the affine Lie algebra $D_n^{(1)}$ in the next section.

3. Bilinear equations for $D_n^{(1)}$

Let $\mathfrak{g}$ be an affine Lie algebra of type $D_n^{(1)}$. In this section we want to construct the two bases (2.3), (2.4) of $\mathfrak{g}$, and then write down the Kac–Wakimoto bilinear equations (2.8). Our result implies that Givental and Milanov's conjecture on the total descendant potential of D_n singularity is true.

Let us consider the corresponding simple Lie algebra first. The simple Lie algebra $\mathring{\mathfrak{g}}$ of type D_n possesses the following $2n$-dimensional matrix realization [5]:

$$\mathring{\mathfrak{g}} = \{A \in \mathbb{C}^{2n\times 2n} \mid A = -SA^T S\}, \qquad S = \sum_{i=1}^n (-1)^{i-1}(e_{ii} + e_{2n+1-i,2n+1-i}). \tag{3.1}$$

Here $e_{i,j}$ is the $2n \times 2n$ matrix that takes value 1 at the (i, j)-entry and zero elsewhere, and $A^T = (a_{l+1-j,k+1-i})$ for any $k \times l$ matrix $A = (a_{ij})$. In this matrix realization, a set of Weyl generators can be chosen as

$$e_i = e_{i+1,i} + e_{2n+1-i,2n-i} \ (1 \leqslant i \leqslant n-1), \qquad e_n = \tfrac{1}{2}(e_{n+1,n-1} + e_{n+2,n}), \tag{3.2}$$

$$f_i = e_{i,i+1} + e_{2n-i,2n+1-i} \ (1 \leqslant i \leqslant n-1), \qquad f_n = 2(e_{n-1,n+1} + e_{n,n+2}), \tag{3.3}$$

$$h_i = -e_{i,i} + e_{i+1,i+1} - e_{2n-i,2n-i} + e_{2n+1-i,2n+1-i} \ (1 \leqslant i \leqslant n-1), \tag{3.4}$$

$$h_n = -e_{n-1,n-1} - e_{n,n} + e_{n+1,n+1} + e_{n+2,n+2}. \tag{3.5}$$

Besides them, we also need the following elements in $\mathring{\mathfrak{g}}$:

$$e_0 = \tfrac{1}{2}(e_{1,2n-1} + e_{2,2n}), \qquad f_0 = 2(e_{2n-1,1} + e_{2n,2}), \tag{3.6}$$

$$h_0 = e_{1,1} + e_{2,2} - e_{2n-1,2n-1} - e_{2n,2n}. \tag{3.7}$$

Recall that the normalized Killing form $(A|B) = \tfrac{1}{2}\operatorname{tr}(AB)$ and the Coxeter number $h = 2n-2$ of $\mathring{\mathfrak{g}}$. We therefore denote the $\mathbb{Z}/h\mathbb{Z}$-principal gradation of $\mathring{\mathfrak{g}}$ as

$$\mathring{\mathfrak{g}} = \bigoplus_{j \in \mathbb{Z}/h\mathbb{Z}} \mathring{\mathfrak{g}}_j;$$

then we have $e_i \in \mathring{\mathfrak{g}}_{\bar{1}}$, $f_i \in \mathring{\mathfrak{g}}_{\overline{-1}}$, $h_i \in \mathring{\mathfrak{g}}_{\bar{0}}$ for $i = 0, \ldots, n$.

Let $\Lambda = \sum_{i=0}^n e_i$ and $\mathring{\mathfrak{s}}$ be the centralizer of Λ in $\mathring{\mathfrak{g}}$. Then $\mathring{\mathfrak{s}}$ is a Cartan subalgebra of $\mathring{\mathfrak{g}}$. We fix a basis $\{T_j | j \in I\}$ of $\mathring{\mathfrak{s}}$ as

$$T_j = \Lambda^j, \qquad j = 1, 3, \ldots, 2n-3,$$
$$T_{(n-1)'} = \sqrt{n-1}\,\kappa\big(e_{n,1} - \tfrac{1}{2}e_{n+1,1} - \tfrac{1}{2}e_{n,2n} + \tfrac{1}{4}e_{n+1,2n}$$
$$+ (-1)^n\big(e_{2n,n+1} - \tfrac{1}{2}e_{2n,n} - \tfrac{1}{2}e_{1,n+1} + \tfrac{1}{4}e_{1,n}\big)\big),$$

where $\kappa = 1$ (resp. $\sqrt{-1}$) when n is even (resp. odd), and I is the set of exponents of $\mathring{\mathfrak{g}}$ given by

$$I = \{1, 3, 5, \ldots, 2n-3\} \cup \{(n-1)'\}.$$

Here $(n-1)'$ indicates that when n is even, the multiplicity of the exponent $n-1$ is 2. These matrices T_j belong to $\mathring{\mathfrak{g}}_j$, respectively, and satisfy

$$(T_i | T_{h-j}) = (n-1)\delta_{i,j}.$$

To construct the desired bases, we need the root space decomposition of $\mathring{\mathfrak{g}}$ with respect to $\mathring{\mathfrak{s}}$. Note that the set of eigenvalues of Λ is

$$\{\omega \in \mathbb{C} \mid \omega^h = 1\} \cup \{0\},$$

in which the multiplicity of 0 is 2. We choose the eigenvectors $\eta_\omega, \eta_0, \eta_{0'}$ associated with eigenvalues $\omega, 0$, respectively, as follows:

$$\eta_\omega = \big(\tfrac{1}{2}, \omega^{-1}, \ldots, \omega^{-(n-1)}, \tfrac{1}{2}\omega^{n-1}, \omega^{n-2}, \ldots, \omega, 1\big)^t,$$
$$\eta_0 = \big(-\tfrac{1}{2}\psi_1 + \psi_{2n}\big) + \kappa^{-1}\big(\psi_n - \tfrac{1}{2}\psi_{n+1}\big),$$
$$\eta_{0'} = \big(-\tfrac{1}{2}\psi_1 + \psi_{2n}\big) - \kappa^{-1}\big(\psi_n - \tfrac{1}{2}\psi_{n+1}\big),$$

where ψ_i is the $2n$-dimensional column vector with the ith entry being 1 and all other entries being zero, and $\cdot^t$ is the usual transposition of matrices. These eigenvectors give a common eigenspace decomposition for T_j $(j \in I)$:

$$T_j \eta_\alpha = \alpha^j \eta_\alpha, \qquad j = 1, 3, \ldots, 2n-3,$$
$$T_{(n-1)'} \eta_\alpha = ((-1)^{n-1}\delta_{\alpha,0} + (-1)^n \delta_{\alpha,0'})\sqrt{n-1}\,\eta_\alpha.$$

Introduce a map $\sigma : \mathbb{C}^{2n \times 2n} \to \mathring{\mathfrak{g}}$, $A \mapsto A - SA^T S$, and define the $2n \times 2n$ matrices

$$A_{(\alpha,\beta)} = \sigma\left(\eta_\alpha \eta_{-\beta}^T\right),$$

where α, β are eigenvalues of Λ. These matrices satisfy

$$[T_j, A_{(\alpha,\beta)}] = (\alpha^j + \beta^j) A_{(\alpha,\beta)}, \qquad j = 1, 3, \ldots, 2n-3,$$

$$[T_{(n-1)'}, A_{(\alpha,\beta)}] = ((-1)^n (-\delta_{\alpha,0} + \delta_{\alpha,0'}) + \delta_{\beta,0} - \delta_{\beta,0'}) \sqrt{n-1} A_{\alpha,\beta},$$

from which one can obtain the root space decomposition of $\mathring{\mathfrak{g}}$ with respect to $\mathring{\mathfrak{s}}$.

Now denote by $A_{(\alpha,\beta),j}$ the homogeneous components of $A_{(\alpha,\beta)}$ in $\mathring{\mathfrak{g}}_j$, and fix $\omega = \exp(2\pi i / h)$. One can verify the following relations

$$(A_{(1,\omega^r),0} | A_{(-1,-\omega^s),0}) = -h \delta_{r,s},$$

$$(A_{(1,\omega^r),0} | A_{(-1,\alpha),0}) = 0,$$

$$(A_{(1,\alpha),0} | A_{(-1,\beta),0}) = 2(1 - \delta_{\alpha,\beta}),$$

where $1 \leqslant r, s \leqslant n-2$ and $\alpha, \beta \in \{0, 0'\}$. According to these relations, we choose two bases of $\mathring{\mathfrak{g}}$:

$$\{T_j \mid j \in I\} \cup \left\{\tilde{X}_m^{(r)} \mid r = 1, \ldots, n; \ m \in \mathbb{Z}/h\mathbb{Z}\right\},$$

$$\{T_j \mid j \in I\} \cup \left\{\tilde{Y}_m^{(r)} \mid r = 1, \ldots, n; \ m \in \mathbb{Z}/h\mathbb{Z}\right\},$$

	$1 \leqslant r \leqslant n-2$	$r = n-1$	$r = n$
$\tilde{X}_m^{(r)} :$	$\frac{1}{\sqrt{h}} A_{(1,\omega^r),m}$	$\frac{1}{\sqrt{2}} A_{(1,0),m}$	$\frac{1}{\sqrt{2}} A_{(1,0'),m}$
$\tilde{Y}_m^{(r)} :$	$-\frac{1}{\sqrt{h}} A_{(-1,-\omega^r),m}$	$\frac{1}{\sqrt{2}} A_{(-1,0'),m}$	$\frac{1}{\sqrt{2}} A_{(-1,0),m}$

(3.8)

The above two bases of $\mathring{\mathfrak{g}}$ help us to construct a pair of dual bases (2.3), (2.4) of the affine Lie algebra $\mathfrak{g}$ that satisfy (2.5)–(2.7). We use the principal realization of $\mathfrak{g}$ [17]:

$$\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \lambda^m \mathring{\mathfrak{g}}_{\bar{m}} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Note that the set of exponents of $\mathfrak{g}$ is $E = I + h\mathbb{Z}$, and the principal Heisenberg subalgebra is generated by

$$H_j = \sqrt{2} \lambda^j T_{\bar{j}}, \qquad j \in E.$$

The two bases (2.3), (2.4) of $\mathfrak{g}$ can be chosen as

$$\frac{1}{\sqrt{h}} H_j, X_m^{(r)} = \lambda^m \tilde{X}_{\bar{m}}^{(r)}, c, d;$$

$$\frac{1}{\sqrt{h}} H_{-j}, Y_{-m}^{(r)} = \lambda^{-m} \tilde{Y}_{-m}^{(r)}, d, c,$$

with the coefficients $\beta_{r,j} (j \in I)$ that appear in (2.6) given by

$$\beta_{r,j} = \begin{cases} \sqrt{2}(1 + \omega^{rj}), & r = 1, 2, \ldots, n-2, \quad j \neq (n-1)', \\ \sqrt{2}, & r = n-1, n, \quad j \neq (n-1)', \\ \sqrt{2n-2}(\delta_{r,n-1} - \delta_{r,n}), & j = (n-1)'. \end{cases}$$

(3.9)

To write down the Kac–Wakimoto bilinear equations (2.8), we still need to compute the constants $g_r = \left(\mathring{\rho}^\vee \big| \mathring{X}_0^{(r)}\right)\left(\mathring{\rho}^\vee \big| \mathring{Y}_0^{(r)}\right)$. Note that in the principal realization of $\mathfrak{g}$, the Weyl generators are given by

$$\tilde{e}_i = \lambda e_i, \qquad \tilde{f}_i = \lambda^{-1} f_i, \qquad \alpha_i^\vee = h_i + \frac{c}{h}, \qquad i = 0, \ldots, n;$$

so we have

$$\left(\mathring{\rho}^\vee \big| \mathring{X}_0^{(r)}\right) = \left(\mathring{\rho}^\vee \bigg| X_0^{(r)} + \frac{c}{h}\sum_{i=1}^n a_i\right) = \sum_{i=1}^n a_i,$$

where a_i are the coefficients in the following linear expansion:

$$X_0^{(r)} = \sum_{i=1}^n a_i h_i = \sum_{i=1}^n a_i\left(\alpha_i^\vee - \frac{c}{h}\right) \in \mathring{\mathfrak{g}}_0.$$

According to the realization (3.2)–(3.5), given any

$$\mathrm{diag}(b_1, b_2, \ldots, b_{2n}) = \sum_{i=1}^n a_i h_i \in \mathring{\mathfrak{g}}_0,$$

the summation $\sum_{i=1}^n a_i$ reads

$$\sum_{i=1}^n a_i = -\sum_{i=1}^{n-1}(n-i)b_i.$$

By using this formula, we obtain

$$g_r = \begin{cases} \dfrac{n-1}{2}\dfrac{2-\omega^r-\omega^{-r}}{2+\omega^r+\omega^{-r}}, & r = 1,\ldots,n-2, \\[2mm] \dfrac{(n-1)^2}{2}, & r = n-1, n. \end{cases} \tag{3.10}$$

Proposition 3.1. *The constants g_r and $\beta_{r,j}$ in the Kac–Wakimoto hierarchy of bilinear equations (2.8) for $D_n^{(1)}$ are given by (3.9) and (3.10).*

Note that the values $\beta_{r,j}$ depend on the choice of the dual bases (2.3), (2.4). However, it is easy to see that the constants g_r are independent of the choice of such bases.

In [13], Givental and Milanov proved that the total descendant potential for semisimple Frobenius manifolds associated with a simple singularity satisfies the following hierarchy of Hirota bilinear equations:

$$\mathrm{res}_{z=0}z^{-1}\sum_{r=1}^n g_r\, e^{\sum_{j\in E_+} 2\beta_{r,j} z^j y_j}\, e^{-\sum_{j\in E_+}\beta_{r,-j} z^{-j}\partial_{y_j}/j}\tau(t+y)\tau(t-y)$$

$$= \left(2h\sum_{j\in E_+} j\, y_j\partial_{y_j} + \frac{nh(h+1)}{12}\right)\tau(t+y)\tau(t-y), \tag{3.11}$$

where the coefficients $\beta_{r,j}$ are the same as in (2.8), and g_r are given explicitly in [13]. By comparing the constants g_r (3.10) with those in [13], we obtain the following corollary.

Corollary 3.2. *The hierarchy (3.11) for the D_n singularity coincides with the Kac–Wakimoto hierarchy of type $D_n^{(1)}$ associated with the basic representation and its principal vertex operator construction.*

Namely, we conform Givental and Milanov's conjecture [13] for the case D_n.

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4. Concluding remarks

We study in [19] the tau structure of the Drinfeld–Sokolov hierarchy associated with the Kac–Moody algebra $D_n^{(1)}$ and the zeroth vertex of its Dynkin diagram following the approach of [7]. So we can define the tau function by using the tau symmetry of the Hamiltonian structures, and establish the equivalence between this definition of the tau function for this hierarchy and that given by Hollowood and Miramontes [15]. Based on the tau structure, we plan to show that this Drinfeld–Sokolov hierarchy coincides with the biHamiltonian integrable hierarchy constructed according to the axiomatic scheme developed by Dubrovin and Zhang [7, 8] on the formal loop space of the semisimple Frobenius manifold associated with the D_n-type Weyl group. This assertion together with the result of this paper would imply that Givental's total descendant potential associated with the D_n singularity is a tau function of Dubrovin and Zhang's hierarchy.

　　While we prepared to do an analogous computation for the cases E_7, E_8 of Givental and Milanov's conjecture [13], we learned from [9] that Frenkel, Givental and Milanov have obtained a proof of this conjecture in general. We hope, however, that this short paper might be helpful to a better understanding of the relationship between Givental's total descendant potentials and integrable systems.

　Note added in proof.　Frenkel, Givental and Milanov's proof is contained in the preprint [20] which appeared after the submission of this paper; it is based on an explicit computation of the operator product expansions of the related vertex operators.

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